Number Theoretic Functions

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Contents

1 Multiplicative Function

Definition 1.1 (Multiplicative Function)**.** *A number-theoretic function f is* ***multiplicative*** *if $f(1) = 1$ and $f(mn) = f(m)f(n)$, $\forall m, n \in \mathbb{N}$ such that $gcd(m, n) = 1$. Additionally, f is called* ***completely multiplicative*** *if $f(mn) = f(m)f(n)$, $\forall m, n \in \mathbb{N}$.*

Examples.

- $1(n) = 1$: The constant function.
- $Id(n) = n$: The identity function.
- $\varepsilon(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$: The unit function
- $\tau(n) = \sum\limits_{d|n} 1$: The number of divisors function.
- $\sigma(n) = \sum\limits_{d|n} d$: The sum of divisors function.
- $\phi(n) = \sum\limits_{i=1}^{n} [gcd(i, n) = 1]$: Euler's Totient Function (here the third brackets serve as a boolean function, which returns 1 if the condition is true, 0 otherwise.)

Note that $1, Id, \varepsilon$ are completely multiplicative as well, while ϕ, τ, σ aren't.

Theorem 1. *If f is a multiplicative function and if $n = \prod\limits_{i=1}^{r} {p_i}^{e_i}$, then $f(n) = \prod\limits_{i=1}^{r} f({p_i}^{e_i})$.*

Proof. Since $gcd({p_i}^{e_i}, {p_j}^{e_j}) = 1$, $\forall i \neq j$, induction on r proves the theorem. □

2 Dirichlet Convolution

Definition 2.1 (Dirichlet Convolution)**.** *If f and g are two arithmetic functions, then their* ***dirichlet convolution****, denoted by $f * g$, is defined as*

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d)$$

Alternatively, we can write $(f * g)(n) = \sum\limits_{ab=n} f(a)g(b)$.

Properties of Dirichlet Convolution

- Convolution is Commutative: $f * g = g * f$
- It is Associative: $(f * g) * h = f * (g * h)$

  Proof. $((f * g) * h)(n) = \sum\limits_{dc=n} (f * g)(d)h(c) = \sum\limits_{dc=n} \sum\limits_{ab=d} f(a)g(b)h(c) = \sum\limits_{abc=n} f(a)g(b)h(c)$

  Similarly, $(f * (g * h))(n) = \sum\limits_{ad=n} f(a)(g * h)(d) = \sum\limits_{ad=n} f(a) \sum\limits_{bc=d} g(b)h(c) = \sum\limits_{abc=n} f(a)g(b)h(c)$ □

- $f * \varepsilon = f$

  Proof.
  $$(f * \varepsilon)(n) = \sum_{d|n} f(d)\varepsilon(n/d)$$

  Now if $d < n$, $n/d > 1$, so $\varepsilon(n/d) = 0$. Therefore
  $$(f * \varepsilon)(n) = f(n)\varepsilon(1) = f(n)$$

  □

Theorem 2. *if f and g are both multiplicative, so is $f * g$.*

Proof. Let $h = f * g$. Now, if $gcd(m, n) = 1$,

$$h(mn) = \sum_{d|mn} f(d)g(mn/d)$$

Now since $d|mn$ and $gcd(m, n) = 1$, $d = ab$ where $a|m, b|n$ and $gcd(a, b) = 1$. Hence,

$$\begin{aligned} h(mn) &= \sum_{a|m,b|n} f(ab)g(mn/ab) \\ &= \sum_{a|m,b|n} f(a)f(b)g(m/a)g(n/b) \\ &= \sum_{a|m} f(a)g(m/a) \sum_{b|n} f(b)g(n/b) \\ &= h(m)h(n) \end{aligned}$$

□

Corollary 1. *if f is a multiplicative function, and $F(n) = \sum_{d|n} f(n)$, F is multiplicative as well.*

Proof. $F(n) = \sum\limits_{d|n} f(n) = \sum\limits_{d|n} f(n)1(\frac{n}{d}) = (f * 1)(n)$.
Since $f, 1$ both are multiplicative, by **theorem 2**, F is also multiplicative. □

Theorem 3. *if $h = f * g$ and h, g are both multiplicative, so is f.*

Proof. Suppose f is not multiplicative. So, there exists a pair of positive integers (m, n) with $gcd(m, n) = 1$ such that $f(mn) \neq f(m)f(n)$. We take such a pair with smallest mn.

If $mn = 1$ then $f(1) \neq f(1)f(1)$ which implies $f(1) \neq 1$. Now since g is multiplicative, $g(1) = 1$. But $h(1) = (f*g)(1) = f(1)g(1) \neq 1$ since $f(1) \neq 1$. This is a contradiction since h is multiplicative and $h(1) = 1$.

If $mn > 1$ then

$$\begin{aligned}
h(mn) &= \sum_{d|mn} f(d)g(mn/d) \\
&= \sum_{a|m,b|n} f(ab)g(mn/ab) \\
&= \sum_{a|m,b|n,ab<mn} f(a)f(b)g(m/a)g(n/b) + f(mn)g(1) \\
&= \sum_{a|m,b|n} f(a)f(b)g(m/a)g(n/b) - f(m)f(n) + f(mn) \\
&= \sum_{a|m} f(a)g(m/a) \sum_{b|n} f(b)g(n/b) - f(m)f(n) + f(mn) \\
&= h(m)h(n) - f(m)f(n) + f(mn)
\end{aligned}$$

Now if h is multiplicative, $f(m)f(n) = f(mn)$, which implies f is multiplicative as well. □

Definition 2.2 (Dirichlet Inverse)**.** *If f is an arithmetic function, we call g* ***dirichlet inverse*** *of f, denoted by f^{-1}, if $f * g = \varepsilon$.*

Recall that ε is the unit function: $\varepsilon(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$

Theorem 4. *If f is an arithmetic function where $f(1) \neq 0$, then f^{-1} exists.*

Proof. Let $g = \begin{cases} \frac{1}{f(1)} & \text{if } n = 1 \\ -\frac{\sum_{d|n,d<n} f(n/d)g(d)}{f(1)} & \text{if } n > 1 \end{cases}$
Clearly, $(f * g)(1) = 1$. Now for $n > 1$,

$$\begin{aligned}
(f * g)(n) &= (g * f)(n) \\
&= \sum_{d|n} g(d)f(n/d) \\
&= g(n)f(1) + \sum_{d|n,d<n} g(d)f(n/d)
\end{aligned}$$

Now,

$$g(n) = -\frac{\sum_{d|n,d<n} g(d)f(n/d)}{f(1)} \implies -g(n)f(1) = \sum_{d|n,d<n} g(d)f(n/d)$$

So, $(f * g)(n) = g(n)f(1) - g(n)f(1) = 0$. Hence, $(f * g) = \varepsilon$, thus $g = f^{-1}$. □

Corollary 2. *If f is multiplicative, so is f^{-1}*

Proof. Since $\varepsilon = f * f^{-1}$, and both ε and f ar multiplicative, by **theorem 3**, f^{-1} is also multiplicative. □

3 More on common multiplicative functions

3.1 Number-of-Divisors Function

Let $\tau(n)$ denote the number of divisors of n. So, $\tau(n) = \sum_{d|n} 1$. It is easy to see that τ is multiplicative, since $\tau(n) = \sum_{d|n} 1 = \sum_{d|n} 1(d)1(n/d) = (1 * 1)(n)$. Recall $1(n)$ is the constant function, which is completely multiplicative. Hence, by **theorem 2**, τ is also multiplicative.
Now it is easy to derive the formula of this function. What is $\tau(p^x)$ where p is prime? Of course the divisors of p^x are $1, p, p^2, \dots, p^x$, so total (x + 1) divisors. Now for any $n = \prod_{i=1}^{r} {p_i}^{e_i}$, we can simply apply **theorem 1** and get $\tau(n) = \prod_{i=1}^{r}(e_i + 1)$. $\tau(n)$ can be calculated with a simple loop in $O(\sqrt{n})$.

3.2 Sum-of-Divisors Function

The sum of divisors function is denoted by σ. We can write $\sigma(n) = \sum_{d|n} d$. Similar to the previous function, we can write $\sigma(n) = \sum_{d|n} d = \sum_{d|n} Id(d)1(n/d) = (Id * 1)(n)$. Since Id and 1 are both multiplicative, by **theorem 2**, σ is also multiplicative.
Now, note that for a prime p, $\sigma(p^x) = 1 + p + p^2 + \cdots + p^x = \frac{p^{x+1}-1}{p-1}$. So, for $n = \prod_{i=1}^{r} {p_i}^{e_i}$, $\sigma(n) = \prod_{i=1}^{r} \frac{{p_i}^{e_i+1}-1}{p_i-1}$.

3.3 Möbius Function

Definition 3.1. *For a positive integer n, the **möbius function**, denoted by μ, is defined as,*

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } p^2|n \text{ for some prime } p \\ (-1)^r & \text{if } n = p_1p_2\dots p_r, \text{ where } p_i \text{ are distinct primes} \end{cases}$$

The next theorem shows a very important property of möbius function.

Theorem 5. $\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases} = \varepsilon(n)$

Proof. For $n = 1$, $\sum_{d|n} \mu(d) = \mu(1) = 1$
When $n > 1$, let $n = \prod_{i=1}^{r} {p_i}^{e_i}$. If $d|n$, then $d = \prod_{i=1}^{r} {p_i}^{f_i}$, where $f_i \leq e_i$, for $1 \leq i \leq r$. Now if $\exists i$ such that $f_i > 1$, $\mu(d)$ will be 0. So, the only divisors we care about are the ones with $f_i \leq 1, \forall i$. This means we have r distinct primes, and we will go through each possible combinations and add their contribution to the sum accordingly. For example, if d contains k primes, then we have $\binom{r}{k}$

possible values for d, and each of them will contribute $(-1)^k$ to the sum. So,

$$\begin{aligned}
\sum_{d|n} \mu(d) &= \sum_{\substack{(f_1,f_2,\ldots,f_r)\\ f_i\in\{0,1\}}} \mu(\prod_{i=1}^{r} {p_i}^{f_i}) \\
&= (-1)^0\binom{r}{0} + (-1)^1\binom{r}{1} + \cdots + (-1)^r\binom{r}{r} \\
&= \sum_{i=0}^{r}\binom{r}{i}(-1)^i \\
&= (1-1)^r \\
&= 0
\end{aligned}$$

□

We also conclude that μ is **multiplicative** from this theorem, since,

$$\sum_{d|n} \mu(d) = \sum_{d|n} \mu(d)1(n/d) = (\mu * 1)(n) = \varepsilon(n)$$

This implies μ is the dirichlet inverse of the constant function. So, by **Corollary 2**, μ is multiplicative. Since μ is the dirichlet inverse of the constant function, using the definition of g in **theorem 4**, we can write that,

$$\mu(n) = \sum_{d|n, d<n} -\mu(d)$$

Now we can use this in sieve to calculate the value of möbius function for integers from 1 to n. Here's an implementation in C++:

```
mu[1] = 1;
for(int i = 1; i < N; i++) {
    for(int j = 2 * i; j < N; j+=i) {
        mu[j] -= mu[i];
    }
}
```

This is basically the standard sieve: instead of iterating over d for each n, for each d, we loop over its multiples and add its contribution to the answer of the multiples. So, it runs in $O(n \log n)$ time.

3.4 Euler's Totient Function

Definition 3.2. *For a positive integer n, **Euler's Totient Function**, denotes as $\phi(n)$, is defined as the number of positive integers i up to n such that $gcd(i, n) = 1$.*

For example, $\phi(4) = 2$, since from 1 to 4, only 1 and 3 are coprimes with 4.

Theorem 6. $\sum_{d|n} \phi(d) = n$

Proof. Let S_d be the set of all positive integers up to n whose gcd with n is d. Now, $gcd(n, m) = d$ implies $gcd(n/d, m/d) = 1$. By definition, the number of such m is $\phi(n/d)$. So,

$$\sum_{d|n} |S_d| = \sum_{d|n} \phi(n/d)$$

Clearly, the sets S_d for all d are pairwise disjoint. Moreover, for every integer $1 \leq k \leq n$, there exists a d such that $d|n$ and $k \in S_d$. Hence, $\sum_{d|n} |S_d| = \sum_{d|n} \phi(n/d) = \sum_{d|n} \phi(d) = n$. ☐

Since, $\sum_{d|n} \phi(d) = \sum_{d|n} \phi(d)1(n/d) = (\phi * 1)(n) = Id(n)$, we also see that ϕ is a multiplicative function.
Note that if p is prime, $\phi(p) = p - 1$, and $\phi(p^k) = p^k - p^{k-1} = p^{k-1}(p-1)$. So, if $n = \prod_{i=1}^{r} {p_i}^{e_i}$, by **theorem 1**, we get

$$\phi(n) = \prod_{i=1}^{r} {p_i}^{e_i - 1}(p_i - 1) = \prod_{i=1}^{r} {p_i}^{e_i} \frac{(p_i - 1)}{p_i} = n \prod_{i=1}^{r} (1 - \frac{1}{p_i})$$

which is the well-known formula for calculating $\phi(n)$.
About implementation, we can simply initialize $\phi(n) = n$, and for each prime divisor p, we update its value by multiplying it with $(1 - \frac{1}{p}) = \frac{p-1}{p}$. Here's the code:

```
for (int i = 1; i <= n; i++) phi[i] = i;
for(int p = 2; p <= n; p++) {
  if (phi[p] == p) {
    phi[p] = p - 1;
    for (int i = 2 * p; i <= n; i += p) {
      phi[i] = (phi[i] / p) * (p - 1);
    }
  }
}
```

4 The Möbius Inversion

Theorem 7. *Let f and g be two arithmetic functions. Then, $g(n) = \sum_{d|n} f(d)$ if and only if $f(n) = \sum_{d|n} g(d)\mu(n/d)$.*

Proof. Note that $g(n) = \sum_{d|n} f(d) = \sum_{d|n} f(d)1(n/d) = (f * 1)(n)$, which implies $g = f * 1$. Similarly, $f(n) = \sum_{d|n} g(d)\mu(n/d) = (g * \mu)(n)$, which implies $f = g * \mu$. Hence it suffices to show that $g = f * 1$ if and only if $f = g * \mu$.

$$g * \mu = (f * 1) * \mu = f * (1 * \mu) = f * \varepsilon = f$$

. Now for converse,

$$f * 1 = (g * \mu) * 1 = g * (1 * \mu) = g * \varepsilon = g$$

. ☐

I kept this single theorem in a separate section only because it is probably the most useful part of the entire note!

5 Problems

Problem 1. Given n, calculate the number of pairs (i, j) such that $i, j \in [1, n]$ and $gcd(i, j) = 1$.
Solution. We basically need to find the value of $\sum_{i=1} \sum_{j=1} [gcd(i, j) = 1]$. Now note that $[gcd(i, j) = 1] = \varepsilon(gcd(i, j))$. Hence,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}[gcd(i,j)=1] = \sum_{i=1}^{n}\sum_{j=1}^{n}\varepsilon(gcd(i,j)) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{d|gcd(i,j)}\mu(d)$$

Now, $d|gcd(i,j)$ means $d|i$ and $d|j$. So,

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{d|gcd(i,j)}\mu(d) &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{d=1}^{n}[d\mid i][d\mid j]\mu(d)\\ &= \sum_{d=1}^{n}\mu(d)\sum_{i=1}^{n}[d\mid i]\sum_{j=1}^{n}[d\mid j]\\ &= \sum_{d=1}^{n}\mu(d)(\lfloor\frac{n}{d}\rfloor)^2\end{aligned}$$

Note that the number of multiples of d from 1 to n is $\lfloor\frac{n}{d}\rfloor$. We can calculate the value of $\mu(i)$ for integers up to n with sieve, and calculate the final answer with a simple loop.
The sequence $\lfloor\frac{n}{1}\rfloor, \lfloor\frac{n}{2}\rfloor, \ldots, \lfloor\frac{n}{n}\rfloor$ contains at most $2\sqrt{n}$ unique values, so if we precalculate the $\mu(i)$, we can calculate the answer for each n in $O(\sqrt{n})$ time.

```
for (int l = 1, r; l <= n; l = r + 1) {
  r = n / (n / i);
  //now for all l <= i <= r, n / i is equal.
}
```

Problem 2. Given n, calculate the number of pairs (i,j) such that $1 \le i < j \le n$ and $gcd(i,j)=1$.
Solution. We need to calculate $\sum_{i=1}^{n}\sum_{j=i+1}^{n}[gcd(i,j)=1]$.

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=i+1}^{n}[gcd(i,j)=1] &= \sum_{j=2}^{n}\sum_{i=1}^{j-1}[gcd(i,j)=1]\\ &= \sum_{j=2}^{n}\phi(j)\end{aligned}$$

Now it's a simple loop!
Problem 3. Given a, b, k, calculate the number of pairs (x,y) such that $x \in [1,a], y \in [1,b]$ and $gcd(x,y)=k$. Also, (x,y) and (y,x) are considered the same.
Solution. First of all, note that $gcd(x,y)=k$ implies that $gcd(x/k, y/k)=1$. So, we can rephrase the statement: calculate the number of pairs (x,y) such that $x \in [1,\lfloor\frac{a}{k}\rfloor], y \in [1,\lfloor\frac{b}{k}\rfloor]$ and $gcd(x,y)=1$.
Let $n=\lfloor\frac{a}{k}\rfloor$ and $m=\lfloor\frac{b}{k}\rfloor$. For simplicity, let's assume $n \ge m$ (otherwise we can simply swap n, m). We need to calculate $\sum_{i=1}^{n}\sum_{j=1}^{min(m,i)}[gcd(i,j)=1]$.

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{min(m,i)}[gcd(i,j)=1] &= \sum_{i=1}^{m}\sum_{j=1}^{i}[gcd(i,j)=1] + \sum_{i=m+1}^{n}\sum_{j=1}^{m}[gcd(i,j)=1] \\
&= \sum_{i=1}^{m}\phi(i) + \sum_{i=m+1}^{n}\sum_{j=1}^{m}\sum_{d|gcd(i,j)}\mu(d) \\
&= \sum_{i=1}^{m}\phi(i) + \sum_{i=m+1}^{n}\sum_{j=1}^{m}\sum_{d|i,d|j}\mu(d) \\
&= \sum_{i=1}^{m}\phi(i) + \sum_{i=m+1}^{n}\sum_{j=1}^{m}\sum_{d=1}^{m}[d \mid i][d \mid j]\mu(d) \\
&= \sum_{i=1}^{m}\phi(i) + \sum_{d=1}^{m}\mu(d)\sum_{i=m+1}^{n}[d \mid i]\sum_{j=1}^{m}[d \mid j] \\
&= \sum_{i=1}^{m}\phi(i) + \sum_{d=1}^{m}\mu(d)(\lfloor\frac{n}{d}\rfloor - \lfloor\frac{m}{d}\rfloor)(\lfloor\frac{m}{d}\rfloor)
\end{aligned}$$

Now you can calculate it with a simple loop in $O(m)$.

Problem 4. Given n, calculate $\sum_{i=1}^{n}\sum_{j=1}^{n} gcd(i,j)$.
Solution. From **theorem 6**, we know that $\sum_{d|n}\phi(d) = n$. So,

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} gcd(i,j) &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{d|gcd(i,j)}\phi(d) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{d=1}^{n}[d \mid i][d \mid j]\phi(d) \\
&= \sum_{d=1}^{n}\phi(d)(\sum_{i=1}^{n}[d \mid i])(\sum_{j=1}^{n}[d \mid j]) \\
&= \sum_{d=1}^{n}\phi(d)\lfloor\frac{n}{d}\rfloor^2
\end{aligned}$$

which can be calculated in $O(\sqrt{n})$ for each n after the sieve (which would take $O(n\log n)$ or $O(n)$ if you use linear sieve).

Problem 5. (Hyperrectangle GCD)Let there be a k-dimensional hyperrectangle with dimensions $n_1, n_2, \ldots n_k$. Each unit cell of the hyperrectangle is addressed by $(p_1, p_2, \ldots, p_k)$ where $1 \le p_i \le n_i$ for all $i \le k$. The value of a cell is the **gcd** of its coordinates. Calculate the sum of value over all cells of the hyperrectangle.
Solution. We are basically asked to calculate the value of:

$$\sum_{x_1=1}^{n_1}\sum_{x_2=1}^{n_2}\cdots\sum_{x_k=1}^{n_k} gcd(x_1, x_2, \ldots, x_k)$$

Let $cnt[x]$ be the number of such tuples whose gcd is x. Also, let $n = \min_{i=1}^{k} n_i$. Then our answer will be $\sum_{g=1}^{n} g \times cnt[g]$. We can write $g = \sum_{d|g} \phi(d)$. So,

$$\sum_{g=1}^{n} g \times cnt[g] = \sum_{g=1}^{n} \sum_{d|g} \phi(d) \times cnt[g]$$

Instead of iterating over divisors d of each g, we can iterate over multiples g of each d. So,

$$\begin{aligned}\sum_{g=1}^{n} g \times cnt[g] &= \sum_{g=1}^{n} \sum_{d|g} \phi(d) \times cnt[g] \\ &= \sum_{d=1}^{n} \phi(d) \sum_{d|g} cnt[g]\end{aligned}$$

Now, $\sum_{d|g} cnt[g]$ is basically the number of tuples whose gcd is divisible by d. It is easy to see that the number of such tuples are $\lfloor \frac{n_1}{d} \rfloor \times \lfloor \frac{n_2}{d} \rfloor \times \cdots \times \lfloor \frac{n_k}{d} \rfloor$. Hence the answer is,

$$\sum_{d=1}^{n} \phi(d) (\lfloor \frac{n_1}{d} \rfloor \times \lfloor \frac{n_2}{d} \rfloor \times \cdots \times \lfloor \frac{n_k}{d} \rfloor)$$

which is calulatable in $O(nk)$.

Problem 6. (Devu and Birthday Celebration) Answer Q ($Q \le 10^5$) queries: Given n and f ($n, f \le 10^5$), calculate the number of ways to distribute n sweets among f friends so that if i-th friend got a_i sweets, $a_i \ge 1$ for all i and $gcd(a_1, a_2, \dots, a_f) = 1$. Note that the ordering matters: $a = [1, 2]$ and $a = [2, 1]$ are considered different.

Solution. First of all, how many ways are there to distribute n sweets among f people so that everyone gets at least one sweet (ignoring the gcd condition)? For that, we can represent a distribution with a string of **'o'**and **'|'**, where **'o'** means we give a sweet to the current person and **'|'** means we move to the next person. For example, if $n = 6$ and $f = 3$, then **'oo|ooo|o'** denotes $a = \{2, 3, 1\}$. Since the string has n **o**'s,we have $(n-1)$ places to insert $(f-1)$ |'s, which can be done in $\binom{n-1}{f-1}$ ways.

Now for a fixed f, let $h(n)$ be the number of ways to distribute n sweets among f people so that everyone gets at least one sweet. From the explanation above, we get $h(n) = \binom{n-1}{f-1}$. Now let $f(n)$ be the number of ways to distribute n sweets among f people so that each gets at least 1 and gcd of all a_i's is 1 (which is basically the answer of the problem).

For a particular distribution, if $gcd(a_1, a_2, \dots a_f) = d$, then $d|n$.

$$\begin{aligned} & a_1 + a_2 + \cdots + a_f = n \\ \implies & d(a'_1 + a'_2 + \cdots + a'_f) = n \\ \implies & a'_1 + a'_2 + \cdots + a'_f = \frac{n}{d}\end{aligned}$$

where $gcd(a'_1, a'_2, \dots, a'_f) = 1$. Since $a'_1 + a'_2 + \cdots + a'_f = \frac{n}{d}$ and $gcd(a'_1, a'_2, \dots, a'_f) = 1$, by definition, the number of such a' is $f(n/d)$. Hence, we get

$$h(n) = \sum_{d|n} f(n/d) = \sum_{d|n} f(d)$$

Finally, we can apply möbius inversion!

$$f(n) = \sum_{d|n} h(d)\mu(n/d) = \sum_{d|n} \binom{d-1}{f-1}\mu(n/d)$$

We can simply precalculate the $\mu(i)$ and the divisors for integer up to 10^5 to answer the queries efficiently.

Problem 7. (COPRIME3) Given an array a of length n ($1 \le a[i] \le 10^6, 1 \le n \le 10^5$), calculate the number of triplets $i < j < k$ such that $gcd(a[i], a[j], a[k]) = 1$.

Solution. We basically need to calculate $\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=j+1}^{n} [gcd(a[i], a[j], a[k]) = 1]$. So,

$$\begin{aligned}
\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=j+1}^{n} [gcd(a[i], a[j], a[k]) = 1] &= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=j+1}^{n} \varepsilon(gcd(a[i], a[j], a[k])) \\
&= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=j+1}^{n} \sum_{d|gcd(a[i],a[j],a[k])} \mu(d) \\
&= \sum_{d=1}^{MAXV} \mu(d) \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=j+1}^{n} [d|gcd(a[i], a[j], a[k])]
\end{aligned}$$

Now, $d|gcd(a[i], a[j], a[k])$ implies d separately divides each of them.
So, $\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=j+1}^{n} [d|gcd(a[i], a[j], a[k])]$ is basically the number of triplets whose elements are divisible by d. If $cnt[d]$ is the number of elements in the array a divisible by d, then number of such triplets will be $\binom{cnt[d]}{3}$. Calculating $cnt[]$ array is pretty easy: if $freq[x]$ is the number of occurrances of x, then $cnt[d] = \sum_{d|x} freq[x]$. Since we will be iterating over multiples for each d, it will take $O(n \log n)$ time. Hence we get the final answer:

$$\sum_{d=1}^{MAXV} \mu(d) \binom{cnt[d]}{3}$$

Problem 8. (CF Mike and Foam) You are given an array a and q queries. ($|a|, q \le 2 \times 10^5, a[i] \le 5 \times 10^5$). You also have a list L, which is initially empty. In each query, you will be given an integer x. If $x \in L$, then you will remove x from L, otherwise add x to L. Then you need to output the number of pairs (i, j) where $i < j$ and $i, j \in L$ such that $gcd(a_i, a_j) = 1$.

Solution. Exercise. Check my code if needed here.

Problem 9. (CF The Holmes Children) Let $f(n)$ be the number of distinct positive integer pairs (x, y) such that $x + y = n$ and $gcd(x, y) = 1$ (additionally, $f(1) = 1$). Let $g(n) = \sum_{d|n} f(d)$. Given n and k, calculate $F_k(n)$, which is defined as:

$$F_k(n) = \begin{cases} f(g(n)) & \text{if } k = 1 \\ g(F_{k-1}(n)) & \text{if } k > 1 \text{ and } k \bmod 2 = 0 \\ f(F_{k-1}(n)) & \text{if } k > 1 \text{ and } k \bmod 2 = 1 \end{cases}$$

Solution. Exercise.

References

- Multiplicative Function, Wikipedia
- Beginning Number Theory, Neville Robbins
- [Tutorial] Math note — Möbius inversion